SPECIAL RIEMANNIAN GEOMETRIES MODELED ON THE DISTINGUISHED SYMMETRIC SPACES(Developments of Cartan Geometry and Related Mathematical Problems)

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SPECIAL RIEMANNIAN GEOMETRIES MODELED ON THE DISTINGUISHED SYMMETRIC SPACES

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ABSTRACT. We propose studies of special Riemannian geometries with structure groups $H_1 = \mathbf{SO}(3) \subset \mathbf{SO}(5)$, $H_2 = \mathbf{SU}(3) \subset \mathbf{SO}(8)$, $H_3 = \mathbf{Sp}(3) \subset \mathbf{SO}(14)$ and $H_4 = \mathbf{F_4} \subset \mathbf{SO}(26)$ in respective dimensions 5, 8, 14 and 26. These geometries, have torsionless models with symmetry groups $G_1 = \mathbf{SU}(3)$, $G_2 = \mathbf{SU}(3)\times\mathbf{SU}(3)$, $G_3 = \mathbf{SU}(6)$ and $G_4 = \mathbf{E}_6$. The groups H_k and G_k constitute a part of the 'magic square' for Lie groups. Apart from the H_k geometries in dimensions n_k, the 'magic square' Lie groups suggest studies of a finite number of other special Riemannian geometries. Among them the smallest dimensional are $\mathbf{U}(3)$ geometries in dimension 12. The other structure groups for these Riemannian geometries are: $\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))$, $\mathbf{U}(6)$, $\mathbf{E}_6\times\mathbf{SO}(2)$, $\mathbf{Sp}(3)\times\mathbf{SU}(2)$, $\mathbf{SU}(6)\times\mathbf{SU}(2)$, $\mathbf{SO}(12)\times\mathbf{SU}(2)$ and $\mathbf{E}_7\times\mathbf{SU}(2)$. The respective dimensions are: 18, 30, 54, 28, 40, 64 and 112. This list is supplemented by the two 'exceptional' cases of $\mathbf{SU}(2)\times\mathbf{SU}(2)$ geometries in dimension 8 and $\mathbf{SO}(10)\times\mathbf{SO}(2)$ geometries in dimension 32.

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1. MOTIVATION

The motivation for this paper comes from type II B string theory (see e.g. [1]), where one considers $n = 6$-dimensional compact Riemannian manifold (X, g) which, in addition to the Levi-Civita connection ∇^{LC}, is equipped with:

- a *metric* conection ∇^T with *totally skew-symmetric torsion* T,
- a spinor field Ψ on X.

Special Riemannian structure $(X, g, \nabla^T, T, \Psi)$ is supposed to satisfy a number of field equations including:

$$\nabla^T\Psi = 0, \quad \delta(T) = 0, \quad T\cdot\Psi = \mu\Psi, \quad Ric^{\nabla^T} = 0.$$

To construct the solutions for these equations one may proceed as follows.

- Let Υ be an object (e.g. a tensor), whose isotropy under the action of $\mathbf{SO}(n)$ is $H \subset \mathbf{SO}(n)$. Infinitesimaly, such an object determines the inclusion of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{so}(n)$.
- If (X, g) is endowed with such a Υ we can decompose the Levi-Civita connection 1-form $\overset{LC}{\Gamma} \in \mathfrak{so}(n)\otimes\mathbb{R}^n$ onto $\Gamma \in \mathfrak{h}\otimes\mathbb{R}^n$ and the rest:

$$\overset{LC}{\Gamma} = \Gamma + \tfrac{1}{2}T.$$

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This decomposition is of course not unique, but let us, for a moment, assume that we have a way of choosing it.

- Then the first Cartan structure equation $\mathrm{d}\theta + (\Gamma + \frac{1}{2}T) \wedge \theta = 0$ for the Levi-Civita connection $\overset{LC}{\Gamma}$ may be rewritten to the form
$$\mathrm{d}\theta + \Gamma \wedge \theta = -\tfrac{1}{2}T \wedge \theta$$
and may be interpreted as the first structure equation for a *metric* connection $\Gamma \in \mathfrak{h} \otimes \mathbb{R}^n$ with torsion $T \in \mathfrak{so}(n) \otimes \mathbb{R}^n$.
- Curvature of this connection $K \in \mathfrak{h} \otimes \bigwedge^2 \mathbb{R}^n$ is determined via the second structure equation:
$$K = \mathrm{d}\Gamma + \Gamma \wedge \Gamma.$$
- To escape from the ambiguity in the split $\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T$ we impose the type II string theory requirement, that T should be totally skew-symmetric.
- Thus we require that our special Riemannian geometry (X, g, Υ) must admit a split $\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T$ with $T \in \bigwedge^3 \mathbb{R}^n$ and $\Gamma \in \mathfrak{h} \otimes \mathbb{R}^n$.
- Examples are known of special Riemannian geometries (e.g. nearly Kähler geometries in dimension $n = 6$) in which such a requirement uniquely determines Γ and T.

This leads to the following problem: find all geometries (X, g, Υ) admitting the *unique* split
$$\overset{LC}{\Gamma} = \Gamma + \tfrac{1}{2}T \quad \text{with} \quad T \in \bigwedge^3 \mathbb{R}^n \quad \text{and} \quad \Gamma \in \mathfrak{h} \otimes \mathbb{R}^n.$$
In which dimensions n do they exist? What is Υ which reduces $\mathbf{SO}(n)$ to H for them? What are the possible isotropy groups $H \in \mathbf{SO}(n)$?

2. Special geometries $(X, g, \nabla^T, T, \Psi)$

If $T \in \bigwedge^3 \mathbb{R}^n$ was *identically zero*, then since $\mathfrak{h} \otimes \mathbb{R}^n \ni \Gamma = \overset{LC}{\Gamma}$, the ***holonomy group*** of (X, g) would be *reduced* to $H \subset \mathbf{SO}(n)$. *All irreducible* compact Riemannian manifolds (X, g) with the reduced holonomy group are classified by Berger [2]. These are:

- either *symmetric spaces* G/H, with the holonomy group $H \subset \mathbf{SO}(n)$
- or they are contained in the *Berger's list*:

Holonomy group for g	Dimension of X	Type of X	Remarks
$\mathbf{SO}(n)$	n	generic	
$\mathbf{U}(n)$	$2n$, $n \geq 2$	Kähler manifold	Kähler
$\mathbf{SU}(n)$	$2n$, $n \geq 2$	Calabi-Yau manifold	Ricci-flat,Kähler
$\mathbf{Sp}(n)\cdot\mathbf{Sp}(1)$	$4n$, $n \geq 2$	quaternionic Kähler	Einstein
$\mathbf{Sp}(n)$	$4n$, $n \geq 2$	hyperkähler manifold	Ricci-flat,Kähler
$\mathbf{G}_2$	7	$\mathbf{G}_2$ manifold	Ricci-flat
$\mathbf{Spin}(7)$	8	$\mathbf{Spin}(7)$ manifold	Ricci-flat

We may relax $T = 0$ for geometries with H from Berger's theorem in at least two ways:

- relax $T = 0$ condition to $T \in \bigwedge^3 \mathbb{R}^n$ for H from *the Berger's list.* This approach leads e.g. to *nearly* Kähler geometries for $H = U(n)$, special *nonintegrable* $SU(3)$ geometries in dimension 6, special *nonintegrable* G_2 geometries in dimension 7, etc.
- relax $T = 0$ condition to $T \in \bigwedge^3 \mathbb{R}^n$ for H corresponding to the irreducible symmetric spaces G/H from *Cartan's list.*

In this note we focus on the second possibility. Here the simplest case is related to the first entry in Cartan's list of the irreducible symmetric spaces, namely to $G/H = \mathbf{SU}(3)/\mathbf{SO}(3)$. Thus, one considers a $n = 5$-dimensional manifold $X = \mathbf{SU}(3)/\mathbf{SO}(3)$ with the *irreducible* $\mathbf{SO}(3)$ action at each tangent space at every point of X. In such an approach $X = \mathbf{SU}(3)/\mathbf{SO}(3)$ is the *integrable* ($T = 0$) model for the irreducible $\mathbf{SO}(3)$ geometries in dimension 5.

Friedrich [7] asked the following questions: is it possible to have 5-dimensional Riemannian geometries $(X, g, \nabla^T, T, \Psi)$ for which the torsionless model would be $G/H = \mathbf{SU}(3)/\mathbf{SO}(3)$? If so, what is Υ for such geometries?

In a joint work with Bobieński [3] we answered these questions as follows:

- Tensor Υ whose isotropy group under the action of $\mathbf{SO}(5)$ is the irreducible $\mathbf{SO}(3)$ is determined by the following conditions:
  i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (totally *symmetric*)
  ii) $\Upsilon_{ijj} = 0$, (trace-free)
  iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$.
- A 5-dimensional Riemannian manifold (X, g) equipped with a tensor field Υ satisfying conditions i)-iii) and admitting a unique decomposition $\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T$, with $T \in \bigwedge^3 \mathbb{R}^5$ and $\Gamma \in \mathfrak{so}(3) \otimes \mathbb{R}^5$ is called *nearly integrable* irreducible $\mathbf{SO}(3)$ structure.
- We have examples of such geometries. All our examples admit transitive symmetry group (which may be of dimension 8, 6 and 5)
- In particular, we have a 7-parameter family of nonequivalent examples which satisfy

$$\nabla^T \Psi = 0, \quad \delta(T) = 0, \quad T \cdot \Psi = \mu \Psi$$

  i.e. equations of type IIB string theory (but in wrong dimension!). For this family of examples $T \neq 0$ and, at every point of X, we have two 2-dimensional vector spaces of ∇^T-covariantly constant spinors Ψ. Moreover, since for this family $K = 0$, we also have $Ric^{\nabla^T} = 0$.

3. Distinguished dimensions

A natural question is [8]: what are the possible dimensions n in which there exists a tensor Υ satisfying:

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (total *symmetry*)
ii) $\Upsilon_{ijj} = 0$, (*no* trace)
iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$?

In dimension $n = 5$ tensor Υ has the following features [3]:

- Given Υ_{ijk} we consider a 3rd order polynomial $w(a) = \Upsilon_{ijk}a_ia_ja_k$, where $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$. We have:

$$w(a) = 6\sqrt{3}a_1a_2a_3 + 3\sqrt{3}(a_1^2 - a_2^2)a_4 - (\ 3a_1^2 + 3a_2^2 - 6a_3^2 - 6a_4^2 + 2a_5^2\ )a_5$$

- Note that:

$$w(a) = \det\begin{pmatrix} a_5 - \sqrt{3}a_4 & \sqrt{3}a_3 & \sqrt{3}a_2 \\ \sqrt{3}a_3 & a_5 + \sqrt{3}a_4 & \sqrt{3}a_1 \\ \sqrt{3}a_2 & \sqrt{3}a_1 & -2a_5 \end{pmatrix} \tag{3.1}$$

The last observation led Bryant [4] to the following answer to our question from the beginning of this section: if tensor Υ with the properties i)-iii) exists in dimension $n = 5$, then it also exists in dimensions $n = 8$, $n = 14$ and $n = 26$. This is because in addition to the field of the real numbers $\mathbb{R}$, we also have $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

We now change expression (3.1) into

$$w(a) = \det A = \det\begin{pmatrix} a_5 - \sqrt{3}a_4 & \sqrt{3}\alpha_3 & \sqrt{3}\alpha_2 \\ \sqrt{3}\overline{\alpha}_3 & a_5 + \sqrt{3}a_4 & \sqrt{3}\alpha_1 \\ \sqrt{3}\overline{\alpha}_2 & \sqrt{3}\overline{\alpha}_1 & -2a_5 \end{pmatrix} \tag{3.2}$$

where $a_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots n$ and

- for $n = 5$ we have:

$$\begin{aligned} \alpha_1 &= a_1 \\ \alpha_2 &= a_2 \\ \alpha_3 &= a_3 \end{aligned}$$

- for $n = 8$ we have

$$\begin{aligned} \alpha_1 &= a_1 + a_6\mathrm{i} \\ \alpha_2 &= a_2 + a_7\mathrm{i} \\ \alpha_3 &= a_3 + a_8\mathrm{i} \end{aligned}$$

- for $n = 14$ we have:

$$\begin{aligned} \alpha_1 &= a_1 + a_6\mathrm{i} + a_9\mathrm{j} + a_{10}\mathrm{k} \\ \alpha_2 &= a_2 + a_7\mathrm{i} + a_{11}\mathrm{j} + a_{12}\mathrm{k} \\ \alpha_3 &= a_3 + a_8\mathrm{i} + a_{13}\mathrm{j} + a_{14}\mathrm{k} \end{aligned}$$

- for $n = 26$ we have:

$$\begin{aligned} \alpha_1 &= a_1 + a_6\mathrm{i} + a_9\mathrm{j} + a_{10}\mathrm{k} + a_{15}\mathrm{p} + a_{16}\mathrm{q} + a_{17}\mathrm{r} + a_{18}\mathrm{s} \\ \alpha_2 &= a_2 + a_7\mathrm{i} + a_{11}\mathrm{j} + a_{12}\mathrm{k} + a_{19}\mathrm{p} + a_{20}\mathrm{q} + a_{21}\mathrm{r} + a_{22}\mathrm{s} \\ \alpha_3 &= a_3 + a_8\mathrm{i} + a_{13}\mathrm{j} + a_{14}\mathrm{k} + a_{23}\mathrm{p} + a_{24}\mathrm{q} + a_{25}\mathrm{r} + a_{26}\mathrm{s} \end{aligned}$$

with i imaginary unit, $\mathrm{i}, \mathrm{j}, \mathrm{k}$ imaginary quaternion units and $\mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s}$ imaginary octonion units. Remarkably, modulo the action of the $\mathbf{O}(n)$ group, the symbol det in the above expression is well defined by the demand that the Weierstrass formula

$$\det A = \sum_{\pi \in S_3} \mathrm{sgn}\pi\ A_{1\pi(1)}A_{2\pi(2)}A_{3\pi(3)}$$

assumes *real* values.

Now, we have the following theorems [5, 6].

Theorem 1

For each $n = 5, 8, 14$ i 26 tensor Υ given by

$$\Upsilon_{ijk} a_i a_j a_k = w(a) = \det A$$

satisfies i)-iii).

Theorem 2

In dimensions $n = 5$, 8, 14 i 26 tensor Υ reduces the $\mathbf{GL}(n, \mathbb{R})$ group via $\mathbf{O}(n)$ to a subgroup H_n, where:

- for $n = 5$ group H_5 is the irreducible $\mathbf{SO}(3)$ in $\mathbf{SO}(5)$; Here, the torsionless compact model is: $\mathbf{SU}(3)/\mathbf{SO}(3)$
- for $n = 8$ group H_8 is the irreducible $\mathbf{SU}(3)$ in $\mathbf{SO}(8)$; Here, the torsionless compact model is: $\mathbf{SU}(3)$
- for $n = 14$ group H_{14} is the irreducible $\mathbf{Sp}(3)$ in $\mathbf{SO}(14)$; Here, the torsionless model is: $\mathbf{SU}(6)/\mathbf{Sp}(3)$
- for $n = 26$ group H_{26} is the irreducible $\mathbf{F}_4$ in $\mathbf{SO}(26)$; Here, the torsionless compact model is: $\mathbf{E}_6/\mathbf{F}_4$

Theorem 3

- The only dimensions in which conditions i)-iii) have solutions for Υ_{ijk} are $n = 5, 8, 14, 26$.
- Modulo the action of $\mathbf{O}(n)$ all such tensors are given by $\det A$, where A is a 3×3 traceless hermitian matrix with entries in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, for the respective dimensions 5,8,14,26.

Idea of the proofs:

- The theorems follow from Cartan's works on *isoparametric hypersurfaces in spheres* [5, 6].
- A hypersurface S is isoparametric in $\mathbf{S}^{n-1}$ iff all its *principal curvatures* are *constant.*
- Cartan proved that S is isoparametric in

  $$\mathbf{S}^{n-1} = \{a^i \in \mathbb{R}^n \mid (a^1)^2 + (a^2)^2 + ... + (a^n)^2 = 1\}$$

  and has 3 *distinct* principal curvatures iff $S = \mathbf{S}^{n-1} \cap P_c$, where

  $$P_c = \{a^i \in \mathbb{R}^n \mid w(a) = c = const \in \mathbb{R}\}$$

  and $w = w(a)$ is a homogeneous 3rd order *polynomial* in variables (a^i) such that

  $$\begin{array}{ll} \text{cii)} & \Delta w = 0 \\ \text{ciii)} & |\nabla w|^2 = 9 \, [\, (a^1)^2 + (a^2)^2 + ... + (a^n)^2 \,]^2. \end{array}$$

- He reduced the above differential equations for $w = w(a)$ to equations for a certain function with the properties of a function he encountered when solving the problem of finding Riemannian spaces with absolute parallelism. He proved that such function give rise only to $w = w(a)$ given by formula (3.2). Of course, Cartan's conditions cii)-ciii) for the polynomial $w = w(a)$ translate to our conditions ii)-iii) for the corresponding symmetric tensor Υ such that $w(a) = \Upsilon_{ijk} a^i a^j a^k$.

These three theorems lead to the idea [8] of studies of H_k structures in dimensions $n_k = 5, 8, 14, 26$.

Definition

An H_k structure on a n_k-dimensional Riemannian manifold (M, g) is a structure defined by means of a rank 3 tensor field Υ satisfying

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$,
ii) $\Upsilon_{ijj} = 0$,
iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$.

An H_k structure is called *nearly integrable* iff

$$\nabla^{LC}_X \Upsilon(X, X, X) = 0, \qquad \forall X \in \Gamma(TM).$$

4. Nearly integrable H_k structures and characteristic connection

Now a natural question is: what are the necessary and sufficient conditions for a H_k structure to admit a unique decomposition

$$\overset{LC}{\Gamma} = \Gamma + \tfrac{1}{2}T \quad \text{with} \quad \Gamma \in \mathfrak{h}_k \otimes \mathbb{R}^k \quad \text{and} \quad T \in {\textstyle\bigwedge}^3\mathbb{R}^{n_k}?$$

If such a unique decomposition exists, the connection Γ is called *characteristic connection* of the H_k structure.

The answer to the above question is given by the following theorem [8].

Theorem 4

Every H_k structure that admits a characteristic connection must be *nearly integrable.*

Moreover,

- In dimensions 5 and 14 the nearly integrable condition is also sufficient for the existence of the characteristic connection.
- In dimension 8 the spaces $\mathfrak{h}_k \otimes \mathbb{R}^k$ and $\bigwedge^3\mathbb{R}^{n_k}$ have 1-dimensional intersection V_1. In this dimension a sufficient condition for the existence of characteristic connection Γ is that the Levi-Civita connection $\overset{LC}{\Gamma}$ of a nearly integrable $\mathbf{SU}(3)$ structure does not have V_1 components in the $\mathbf{SU}(3)$ decomposition of $\mathfrak{so}(8) \otimes \mathbb{R}^8$ onto the irreducibles.
- In dimension 26 the Levi-Civita connection $\overset{LC}{\Gamma}$ of a nearly integrable $\mathbf{F}_4$ structure may have values in 52-dimensional irreducible representation V_{52} of $\mathbf{F}_4$, which is not present in the algebraic sum of $\mathfrak{f}_4 \otimes \mathbb{R}^k$ and $\bigwedge^3\mathbb{R}^{n_k}$. The sufficient condition for such structures to admit characteristic Γ is that $\overset{LC}{\Gamma}$ has not components in V_{52}.

Definition

The nearly integrable H_k structures described by Theorem 4 are called *restricted* nearly integrable.

Now we discuss what the restricted nearly integrable condition means for a H_k structure:

- If $n_k = 5$ then, out of the *a priori* 50 independent components of the Levi-Civita connection $\overset{LC}{\Gamma}$, the restricted nearly integrable condition excludes 25. Thus, heuristically, the restricted nearly integrable $\mathbf{SO}(3)$ structures constitute 'a half' of all the possible $\mathbf{SO}(3)$ structures in dimension 5.

- If $n_k = 8$ the Levi-Civita connection has 224 components. The restricted nearly integrable condition reduces it to 118.
- For $n_k = 14$ these numbers reduce from 1274 to 658.
- For $n_k = 26$ the reduction is from 8450 to 3952.

To discuss the possible torsion types of the characteristic connection for H_k geometries we need to know that:

- there are *real* irreducible representations of the group $\mathbf{SO}(3)$ in odd dimensions: $1, 3, 5, 7, 9...$
- there are *real* irreducible representations of the group $\mathbf{SU}(3)$ in dimensions: $1, 8, 20, 27, 70...$
- there are *real* irreducible representations of the group $\mathbf{Sp}(3)$ in dimensions: $1, 14, 21, 70, 84, 90, 126, 189, 512, 525...$
- there are *real* irreducible representations of the group $\mathbf{F}_4$ in dimensions: $1, 26, 52, 273, 324, 1053, 1274, 4096, 8424...$

For each of the possible dimensions $n_k = 5, 8, 14, 26$ we denote a possible irreducible j-dimensional representation of the H_k group by ${}^{n_k}V_j$. Then we have the following theorem [8].

Theorem 5

Let (M, g, Υ) be a nearly integrable H_k structure admitting characteristic connection Γ. The H_k irreducible decomposition of the skew symmetric torsion T of Γ is given by:

- $T \in {}^5V_7 \oplus {}^5V_3$, for $n_k = 5$,
- $T \in {}^8V_{27} \oplus {}^8V_{20} \oplus {}^8V_8 \oplus {}^8V_1$, for $n_k = 8$,
- $T \in {}^{14}V_{189} \oplus {}^{14}V_{84} \oplus {}^{14}V_{70} \oplus {}^{14}V_{21}$, for $n_k = 14$,
- $T \in {}^{26}V_{1274} \oplus {}^{26}V_{1053} \oplus {}^{26}V_{273}$, for $n_k = 26$.

Example [8]: SU(3) structures in dimension 8

Among many interesting features of these structures we mention the following:

- We have examples of these structures admitting a characteristic connection with nonzero torsion.
- All our examples admit transitive symmetry group, which can has dimension ≤ 16.
- We have 2-parameter family of examples with transitive symmetry group of dimension 11, with torsion $T \in {}^8V_{27}$ and the Ricci tensor Ric^Γ of the characteristic connection Γ with 2 different constant eigenvalues of multiplicity 5 and 3
- We have another 2-parameter family of examples with transitive symmetry group of dimension 9, with *vectorial* torsion $T \in {}^8V_8$ and with the Ricci tensor Ric^Γ of the characteristic connection Γ with 2 different constant eigenvalues of multiplicity 4 and 4.
- In the decomposition of $\bigwedge^3\mathbb{R}^8$ onto the irreducible components under the action of $\mathbf{SU}(3)$ there exists a 1-dimensional $\mathbf{SU}(3)$ invariant subspace ${}^8V_1$.
- This space, in an orthonormal coframe adapted to the $\mathbf{SU}(3)$ structure, is spanned by a 3-form

$$\psi = \tau_1 \wedge \theta^6 + \tau_2 \wedge \theta^7 + \tau_3 \wedge \theta^8 + \theta^6 \wedge \theta^7 \wedge \theta^8,$$

where (τ_1, τ_2, τ_3) are 2-forms

$$\begin{aligned} \tau_1 &= \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + \sqrt{3}\theta^1 \wedge \theta^5 \\ \tau_2 &= \theta^1 \wedge \theta^3 + \theta^4 \wedge \theta^2 + \sqrt{3}\theta^2 \wedge \theta^5 \\ \tau_3 &= \theta^1 \wedge \theta^2 + 2\theta^4 \wedge \theta^3 \end{aligned}$$

spanning the 3-dimensional irreducible representation ${}^5\bigwedge^2_3 \simeq \mathfrak{so}(3)$ associated with $\mathbf{SO}(3)$ structure in dimension 5.

- The 3-form ψ can be considered in $\mathbb{R}^8$ without any reference to tensor Υ.
- It is remarkable that this 3-form *alone* reduces the $\mathbf{GL}(8, \mathbb{R})$ to the irreducible $\mathbf{SU}(3)$ in the same way as Υ does.
- Thus, in dimension 8, the H_k structure can be defined either in terms of the *totally symmetric* Υ or in terms of the *totally skew symmetric* ψ.
- In this sense the 3-form ψ and the 2-forms (τ_1, τ_2, τ_3) play the same role in the relations between $\mathbf{SU}(3)$ structures in dimension *eight* and $\mathbf{SO}(3)$ structures in dimension *five* as the 3-form

$$\phi = \sigma_1 \wedge \theta^5 + \sigma_2 \wedge \theta^6 + \sigma_3 \wedge \theta^7 + \theta^5 \wedge \theta^6 \wedge \theta^7$$

and the self-dual 2-forms

$$\begin{aligned} \sigma_1 &= \theta^1 \wedge \theta^3 + \theta^4 \wedge \theta^2 \\ \sigma_2 &= \theta^4 \wedge \theta^1 + \theta^3 \wedge \theta^2 \\ \sigma_3 &= \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4 \end{aligned}$$

play in the relations between $\mathbf{G}_2$ structures in dimension *seven* and $\mathbf{SU}(2)$ structures in dimension *four*.

5. The magic square and special geometries modeled on distinguished symmetric spaces

Looking at the torsionless models X_k for the H_k structures given in Theorem 2 we see that each group H_k has its associated group G_k such that $X_k = G_k/H_k$. Remarkably the Lie algebras of the groups H_k and G_k constitute, respectively, the first and the second column of the celebrated 'magic square' of the Lie algebras [9, 10]:

$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathfrak{su}(3)$	$2\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Let G_k, $\mathcal{G}_k$ and $\tilde{\mathcal{G}}_k$ denote the respective compact Lie groups corresponding to the Lie algebras of the second, third and the fourth columns of the magic square. The observation opening this section suggests that the pairs $(G_k, \mathcal{G}_k)$ and $(\mathcal{G}_k, \tilde{\mathcal{G}}_k)$, with the homogeneous spaces $\mathcal{G}_k/G_k$ and $\tilde{\mathcal{G}}_k/\mathcal{G}_k$ may model $T \equiv 0$ cases of other interesting special Riemannian geometries with skew-symmetric torsion. This suggestion should be a bit modified, since the homogeneous spaces $\mathcal{G}_k/G_k$ and $\tilde{\mathcal{G}}_k/\mathcal{G}_k$ are reducible. To have *irreducible* symmetric spaces we need

- either to take the Lie groups $\mathcal{G}_k$ corresponding to the third column of the magic square and divide them by the compact Lie groups corresponding to the Lie algebras from the following table:

$\mathfrak{su}(3) \oplus \mathbb{R}$
$2\mathfrak{su}(3) \oplus \mathbb{R}$
$\mathfrak{su}(6) \oplus \mathbb{R}$
$\mathfrak{e}_6 \oplus \mathbb{R}$

,

- or we need to take the Lie groups $\tilde{\mathcal{G}}_k$ corresponding to the forth column of the magic square and divide them by the compact Lie groups corresponding to the Lie algebras from the following table:

$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$

.

This leads to twelve torsionless models of special Riemannian geometries [8] given in the table below:

$\mathbf{SU}(3)/\mathbf{SO}(3)$	$\mathbf{Sp}(3)/\mathbf{U}(3)$	$\mathbf{F}_4/(\mathbf{Sp}(3) \times \mathbf{SU}(2))$
$\mathbf{SU}(3)$	$\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$	$\mathbf{E}_6/(\mathbf{SU}(6) \times \mathbf{SU}(2)$
$\mathbf{SU}(6)/\mathbf{Sp}(3)$	$\mathbf{SO}(12)/\mathbf{U}(6)$	$\mathbf{E}_7/(\mathbf{SO}(12) \times \mathbf{SU}(2))$
$\mathbf{E}_6/\mathbf{F}_4$	$\mathbf{E}_7/(\mathbf{E}_6 \times \mathbf{SO}(2))$	$\mathbf{E}_8/(\mathbf{E}_7 \times \mathbf{SU}(2))$

.

It is an interesting question if these 12 symmetric spaces can be deformed to obtain twelve classes of special geometries X with totally skew symmetric torsion and the characteristic connection. The dimensions n of X and the structure groups of the characteristic connection for these geometries are given in the table below:

$n = n_k$	Structure group H_k	$n = 2(n_k+1)$	Structure group	$n = 4(n_k+2)$	Structure group
5	$\mathbf{SO}(3)$	12	$\mathbf{U}(3)$	28	$\mathbf{Sp}(3) \times \mathbf{SU}(2)$
8	$\mathbf{SU}(3)$	18	$\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$	40	$\mathbf{SU}(6) \times \mathbf{SU}(2)$
14	$\mathbf{Sp}(3)$	30	$\mathbf{U}(6)$	64	$\mathbf{SO}(12) \times \mathbf{SU}(2)$
26	$\mathbf{F}_4$	54	$\mathbf{E}_6 \times \mathbf{SO}(2)$	112	$\mathbf{E}_7 \times \mathbf{SU}(2)$

A quick look at the Cartan's list of symmetric spaces shows that this list should be supplemented by two exceptional cases:

1) $\dim X = 8$, with the structure group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ and with the torsionless model of compact type $X = \mathbf{G}_2/(\mathbf{SU}(2) \times \mathbf{SU}(2))$.
2) $\dim X = 32$, with the structure group $\mathbf{SO}(10) \times \mathbf{SO}(2)$ and with the torsionless model of compact type $X = \mathbf{E}_6/(\mathbf{SO}(10) \times \mathbf{SO}(2))$

Besides the geometries from the first column of the above table, and besides the exceptional geometries of case 1) above (see [8]), we do not know what objects Υ reduce the $\mathbf{O}(n)$ groups to the structure groups included in the table.

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